

Renormalisation of the non-anticommutativity parameter at two loops

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Abstract

We present evidence that the non-anticommutativity parameter for the $\mathcal{N} = \frac{1}{2}$ supersymmetric $SU(N) \otimes U(1)$ gauge theory is unrenormalised through two loops.

1 Introduction

Deformed quantum field theories have been subject to renewed attention in recent years due to their natural appearance in string theory. Initial investigations focussed on theories on non-commutative spacetime in which the commutators of the spacetime co-ordinates become non-zero. More recently [1–9], non-anticommutative supersymmetric theories have been constructed by deforming the anticommutators of the Grassmann co-ordinates θ^α (while leaving the anticommutators of the $\bar{\theta}^{\dot{\alpha}}$ unaltered). Consequently, the anticommutators of the supersymmetry generators $\bar{Q}_{\dot{\alpha}}$ are deformed while the remainder are unchanged. It can be shown that this structure arises in string theory in a background with a constant graviphoton field strength. A graviphoton background $F_{\alpha\beta}$ couples to the field q_α which is the string worldsheet field corresponding to the supercharge Q_α (and also to its worldsheet conjugate) in Berkovits’ formulation of the superstring [11]. Upon eliminating q and its conjugate using their equations of motion, one obtains an effective contribution to the lagrangian

$$L_{\text{eff}} = \frac{1}{\alpha'^2} F_{\alpha\beta}^{-1} \partial \tilde{\theta}^\alpha \partial \tilde{\theta}^\beta, \quad (1)$$

where $\tilde{\theta}$ is the worldsheet conjugate of θ . This leads to a propagator

$$\langle \theta^\alpha(\tau) \theta^\beta(\tau') \rangle = \alpha'^2 F^{\alpha\beta} \text{sign}(\tau - \tau'). \quad (2)$$

With standard open string coupling arguments, this implies

$$\{\theta^\alpha, \theta^\beta\} = \alpha'^2 F^{\alpha\beta} \equiv C^{\alpha\beta}, \quad (3)$$

where $C^{\alpha\beta}$ is usually referred to as the “non-anticommutativity parameter”. We then find

$$\begin{aligned} \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} &= -4C^{\alpha\beta} \sigma_{\alpha\dot{\alpha}}^\mu \sigma_{\beta\dot{\beta}}^\mu \frac{\partial^2}{\partial y^\mu \partial y^\nu}, \\ y^\mu &= x^\mu + i\theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}}. \end{aligned} \quad (4)$$

(More details of this derivation can be found in Refs. [9, 10].) It is straightforward to construct non-anticommutative versions of ordinary supersymmetric theories by taking the superspace action and replacing the ordinary product by the Moyal \ast -product [10] which implements the non-anticommutativity. Non-anticommutative versions of the Wess-Zumino model and supersymmetric gauge theories have been formulated in four dimensions [10, 12] and their renormalisability discussed [13–18], with explicit computations up to two loops [19] for the Wess-Zumino model and one loop for gauge theories [20–24]. Even more recently, non-anticommutative theories in two dimensions have been constructed [25–29], and their one-loop divergences computed [30, 31]. In Ref. [32] we returned to a closer examination of the non-anticommutative Wess-Zumino model (with a superpotential) in four dimensions, and showed that to obtain correct results for the theory where the auxiliary fields have been eliminated, from the corresponding results for the uneliminated theory, it is necessary to include in the classical action separate couplings for all the terms which may

be generated by the renormalisation process; and in Ref. [33] we extended this analysis to the gauged $U(1)$ case.

There are obstacles to obtaining a renormalisable $\mathcal{N} = \frac{1}{2}$ theory with a trilinear superpotential in the case of adjoint matter (in the case of matter in the fundamental representation, only a mass term is allowed anyway) [24]. The requirements of $\mathcal{N} = \frac{1}{2}$ invariance and renormalisability impose the choice of gauge group $SU(N) \otimes U(1)$ (rather than $SU(N)$ or $U(N)$) [20], [21]. In the adjoint case with a trilinear superpotential, the matter fields must also be in a representation of $SU(N) \otimes U(1)$. The problem is that the superpotential contains terms with different combinations of $SU(N)$ and $U(1)$ chiral fields which mix under $\mathcal{N} = \frac{1}{2}$ supersymmetry, but for which the Yukawa couplings renormalise differently. However, recently an elegant solution to this problem has been found [34] in which the kinetic terms for the $U(1)$ chiral fields are modified, in such a way that the $SU(N)$ and $U(1)$ chiral fields (and consequently their Yukawa couplings) renormalise in exactly the same way. In Ref. [38] we confirmed the conclusions of Ref. [34] in a component version of their superspace calculation.

The results of Refs. [34,38] imply that the non-anticommutativity parameter (C) which specifies the superspace deformation in $\mathcal{N} = \frac{1}{2}$ supersymmetry is unrenormalised at one loop. It is clearly interesting to ask whether this feature persists at higher orders. A full two-loop calculation would be extremely complex, and the results we present here are only partial in two respects. Firstly we only check the renormalisation of one, judiciously chosen, term in the action (of course if different terms in the action required different renormalisations of C , this would represent a violation of $\mathcal{N} = \frac{1}{2}$ supersymmetry); and secondly, we only check the terms in the two-loop renormalisation constant for C which include the Yukawa coupling, omitting the purely gauge-coupling dependent term. Our conclusion is that there are no Yukawa dependent terms in the renormalisation constant for C through two loops, and we consider it likely that C is unrenormalised at this order.

2 The classical adjoint action

In this section we present the classical form of the adjoint $\mathcal{N} = \frac{1}{2}$ action with a superpotential in the component formalism, including the modifications suggested in Ref. [34]. The adjoint action was first introduced in Ref. [12] for the gauge group $U(N)$. However, as we noted in Refs. [20], [21], at the quantum level the $U(N)$ gauge invariance cannot be retained since the $SU(N)$ and $U(1)$ gauge couplings renormalise differently; and we are obliged to consider a modified $\mathcal{N} = \frac{1}{2}$ invariant theory with the gauge group $SU(N) \otimes U(1)$. In the adjoint case with a Yukawa superpotential, it turns out that the matter fields must also be in the adjoint representation of $SU(N) \otimes U(1)$. The classical action with a superpotential

may be written

$$\begin{aligned}
S_0 = & \int d^4x \left\{ e^{AB} \left(-\frac{1}{4} F^{\mu\nu A} F_{\mu\nu}^B - i \bar{\lambda}^A \bar{\sigma}^\mu (D_\mu \lambda)^B + \frac{1}{2} D^A D^B \right) \right. \\
& - \frac{1}{2} i C^{\mu\nu} d^{ABC} e^{AD} F_{\mu\nu}^D \bar{\lambda}^B \bar{\lambda}^C \\
& + \bar{F} F - i \bar{\psi} \bar{\sigma}^\mu D_\mu \psi - D^\mu \bar{\phi} D_\mu \phi + \bar{\phi} D_F \phi + i \sqrt{2} (\bar{\phi} \lambda_F \psi - \bar{\psi} \bar{\lambda}_F \phi) \\
& + C^{\mu\nu} (\sqrt{2} D_\mu \bar{\phi} \bar{\lambda}_D \bar{\sigma}_\nu \psi + i \bar{\phi} F_{D\mu\nu} F) \\
& + (\kappa - 1) [\bar{F}^0 F^0 - i \bar{\psi}^0 \bar{\sigma}^\mu \partial_\mu \psi^0 - \partial^\mu \bar{\phi}^0 \partial_\mu \phi^0 \\
& + d^{000} C^{\mu\nu} (\sqrt{2} \partial_\mu \bar{\phi}^0 \bar{\lambda}^0 \bar{\sigma}_\nu \psi^0 + i \bar{\phi}^0 F_{\mu\nu}^0 F^0) \\
& + d^{ab0} C^{\mu\nu} (\sqrt{2} D_\mu \bar{\phi}^a \bar{\lambda}^b \bar{\sigma}_\nu \psi^0 + i \bar{\phi}^a F_{\mu\nu}^b F^0)] \\
& + \frac{1}{2} \left(y d^{ABC} \phi^A \phi^B F^C - y d^{ABC} \phi^A \psi^B \psi^C + \bar{y} d^{ABC} \bar{\phi}^A \bar{\phi}^B \bar{F}^C - \bar{y} d^{ABC} \bar{\phi}^A \bar{\psi}^B \bar{\psi}^C \right) \\
& + \frac{1}{3} i \bar{y} C^{\mu\nu} f^{abc} D_\mu \bar{\phi}^a D_\nu \bar{\phi}^b \bar{\phi}^c - \frac{1}{3} i \bar{y} C^{\mu\nu} d^{ABE} d^{CDE} F_{\mu\nu}^D \bar{\phi}^A \bar{\phi}^B \bar{\phi}^C \\
& + \kappa_1 \sqrt{2} C^{\mu\nu} d^{abc} (\bar{\phi}^a \bar{\lambda}^b \bar{\sigma}_\nu D_\mu \psi^c + D_\mu \bar{\phi}^a \bar{\lambda}^b \bar{\sigma}_\nu \psi^c + i \bar{\phi}^a F_{\mu\nu}^b F^c) \\
& + \kappa_2 \sqrt{2} C^{\mu\nu} d^{ab0} (\bar{\phi}^0 \bar{\lambda}^a \bar{\sigma}_\nu D_\mu \psi^b + \partial_\mu \bar{\phi}^0 \bar{\lambda}^a \bar{\sigma}_\nu \psi^b + i \bar{\phi}^0 F_{\mu\nu}^a F^b) \\
& + \kappa_3 \sqrt{2} C^{\mu\nu} d^{ab0} (\bar{\phi}^a \bar{\lambda}^b \bar{\sigma}_\nu \partial_\mu \psi^0 + D_\mu \bar{\phi}^a \bar{\lambda}^b \bar{\sigma}_\nu \psi^0 + i \bar{\phi}^a F_{\mu\nu}^b F^0) \\
& + \kappa_4 \sqrt{2} C^{\mu\nu} d^{0ab} (\bar{\phi}^a \bar{\lambda}^0 \bar{\sigma}_\nu D_\mu \psi^b + D_\mu \bar{\phi}^a \bar{\lambda}^0 \bar{\sigma}_\nu \psi^b + i \bar{\phi}^a F_{\mu\nu}^0 F^b) \\
& + \kappa_5 \sqrt{2} C^{\mu\nu} d^{000} (\bar{\phi}^0 \bar{\lambda}^0 \bar{\sigma}_\nu \partial_\mu \psi^0 + \partial_\mu \bar{\phi}^0 \bar{\lambda}^0 \bar{\sigma}_\nu \psi^0 + i \bar{\phi}^0 F_{\mu\nu}^0 F^0) \left. \right\}. \tag{5}
\end{aligned}$$

where

$$\begin{aligned}
\lambda_F &= \lambda^A \tilde{F}^A, & (\tilde{F}^A)^{BC} &= i f^{BAC}, \\
\lambda_D &= \lambda^A \tilde{D}^A, & (\tilde{D}^A)^{BC} &= d^{ABC},
\end{aligned} \tag{6}$$

(similarly for $D_F, F_{D\mu\nu}$), and we have

$$\begin{aligned}
D_\mu \phi &= \partial_\mu \phi + i A_\mu^F \phi, \\
F_{\mu\nu}^A &= \partial_\mu A_\nu^A - \partial_\nu A_\mu^A - f^{ABC} A_\mu^B A_\nu^C,
\end{aligned} \tag{7}$$

with similar definitions for $D_\mu \psi, D_\mu \lambda$. If one decomposes $U(N)$ as $SU(N) \otimes U(1)$ then our convention is that ϕ^a (for example) are the $SU(N)$ components and ϕ^0 the $U(1)$ component. Of course then $f^{ABC} = 0$ unless all indices are $SU(N)$. We note that $d^{ab0} = \sqrt{\frac{2}{N}} \delta^{ab}$, $d^{000} = \sqrt{\frac{2}{N}}$. We also have

$$e^{ab} = \frac{1}{g^2}, \quad e^{00} = \frac{1}{g_0^2}, \quad e^{0a} = e^{a0} = 0. \tag{8}$$

Compared with our previous work such as Ref. [24], we have absorbed a factor of g into our definitions of the fields in the gauge multiplet. We have omitted terms which are $\mathcal{N} = \frac{1}{2}$

supersymmetric on their own (such as terms involving only $\bar{\phi}$, $\bar{\lambda}$ and/or F), which will have no relevance for our current discussion. They were considered in full in Refs. [34]; and indeed we included them ourselves in Refs. [20], [21]. We have, however, included some additional sets of terms (those multiplied by κ_{1-5}) which are required for renormalisability of the theory. Each of these sets of terms is separately $\mathcal{N} = \frac{1}{2}$ invariant.

It is easy to show that Eq. (5) is invariant under

$$\begin{aligned}
\delta A_\mu^A &= -i\bar{\lambda}^A \bar{\sigma}_\mu \epsilon, \\
\delta \lambda_\alpha^A &= i\epsilon_\alpha D^A + (\sigma^{\mu\nu} \epsilon)_\alpha \left[F_{\mu\nu}^A + \frac{1}{2} i C_{\mu\nu} d^{ABC} \bar{\lambda}^B \bar{\lambda}^C \right], \quad \delta \bar{\lambda}_{\dot{\alpha}}^A = 0, \\
\delta D^A &= -\epsilon \sigma^\mu D_\mu \bar{\lambda}^A, \\
\delta \phi &= \sqrt{2} \epsilon \psi, \quad \delta \bar{\phi} = 0, \\
\delta \psi^\alpha &= \sqrt{2} \epsilon^\alpha F, \quad \delta \bar{\psi}_{\dot{\alpha}} = -i\sqrt{2} (D_\mu \bar{\phi}) (\epsilon \sigma^\mu)_{\dot{\alpha}}, \\
\delta F^A &= 0, \\
\delta \bar{F}^A &= -i\sqrt{2} D_\mu \bar{\psi}^A \bar{\sigma}^\mu \epsilon - 2i(\bar{\phi} \epsilon \lambda^F)^A + 2C^{\mu\nu} D_\mu (\bar{\phi}^B \epsilon \sigma_\nu (\bar{\lambda}^D)^{AB}).
\end{aligned} \tag{9}$$

In Eq. (5), $C^{\mu\nu}$ is related to the non-anti-commutativity parameter $C^{\alpha\beta}$ by

$$C^{\mu\nu} = C^{\alpha\beta} \epsilon_{\beta\gamma} \sigma_\alpha^{\mu\nu\gamma}, \tag{10}$$

where

$$\begin{aligned}
\sigma^{\mu\nu} &= \frac{1}{4} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu), \\
\bar{\sigma}^{\mu\nu} &= \frac{1}{4} (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu).
\end{aligned} \tag{11}$$

Our conventions are in accord with [10]; in particular,

$$\sigma^\mu \bar{\sigma}^\nu = -\eta^{\mu\nu} + 2\sigma^{\mu\nu}. \tag{12}$$

Properties of C which follow from Eq. (10) are

$$\begin{aligned}
C^{\alpha\beta} &= \frac{1}{2} \epsilon^{\alpha\gamma} (\sigma^{\mu\nu})_\gamma{}^\beta C_{\mu\nu}, \\
C^{\mu\nu} \sigma_{\nu\alpha\dot{\beta}} &= C_\alpha{}^\gamma \sigma^\mu{}_{\gamma\dot{\beta}}, \\
C^{\mu\nu} \bar{\sigma}_\nu^{\dot{\alpha}\beta} &= -C^\beta{}_\gamma \bar{\sigma}^{\mu\dot{\alpha}\gamma}.
\end{aligned} \tag{13}$$

We use the standard gauge-fixing term

$$S_{\text{gf}} = \frac{1}{2\alpha} \int d^4x e^{AB} (\partial \cdot A)^A (\partial \cdot A)^B \tag{14}$$

with its associated ghost terms. The vector propagator is given by

$$\Delta_{V\mu\nu}^{AB} = -\frac{1}{p^2} \left(\eta_{\mu\nu} + (\alpha - 1) \frac{p_\mu p_\nu}{p^2} \right) (e^{-1})^{AB}. \tag{15}$$

The scalar propagator is

$$\Delta_{\phi}^{AB} = -\frac{1}{p^2} P^{AB} \quad (16)$$

where

$$P^{ab} = \delta^{ab}, \quad P^{00} = \frac{1}{\kappa}, \quad P^{0a} = P^{a0} = 0, \quad (17)$$

the fermion propagator is

$$\Delta_{\psi\alpha\dot{\alpha}}^{AB} = \frac{p_{\mu}\sigma_{\alpha\dot{\alpha}}^{\mu}}{p^2} P^{AB}, \quad (18)$$

where the momentum enters at the end of the propagator with the undotted index, and the auxiliary propagator is

$$\Delta_F^{AB} = P^{AB}. \quad (19)$$

3 Renormalisation

The bare action will be given as usual by replacing fields and couplings by their bare versions, shortly to be given more explicitly. Note that in the $\mathcal{N} = \frac{1}{2}$ supersymmetric case, fields and their conjugates may renormalise differently. We found in Refs. [20], [21] that non-linear renormalisations of λ and \overline{F} were required; and in a subsequent paper [35] we pointed out that non-linear renormalisations of F , \overline{F} are required even in ordinary $\mathcal{N} = 1$ supersymmetric gauge theory when working in the uneliminated formalism. The renormalisations of the remaining fields and couplings are linear as usual (except for κ , κ_{1-5} , see later) and given by (in the case of the $SU(N)$ fields)

$$\begin{aligned} \overline{\lambda}_B^a &= Z_{\lambda}^{\frac{1}{2}} \overline{\lambda}^a, & A_{\mu B}^a &= Z_A^{\frac{1}{2}} A_{\mu}^a, & \phi_B^a &= Z_{\phi}^{\frac{1}{2}} \phi^a, & \psi_B^a &= Z_{\psi}^{\frac{1}{2}} \psi^a, \\ \overline{\phi}_B^a &= Z_{\phi}^{\frac{1}{2}} \overline{\phi}^a, & \overline{\psi}_B^a &= Z_{\psi}^{\frac{1}{2}} \overline{\psi}^a, & g_B &= Z_g g, & y_B &= Z_y y, \\ C_B^{\mu\nu} &= Z_C C^{\mu\nu}, & (\kappa-1)_B &= Z_{\kappa}(\kappa-1), & \kappa_{1-5B} &= Z_{1-5}. \end{aligned} \quad (20)$$

The corresponding $U(1)$ gauge multiplet fields $\overline{\lambda}^0$ etc are unrenormalised; so is g_0 . The renormalisation constants for the $U(1)$ chiral fields will be denoted Z_{ϕ^0} etc and discussed later. In Eq. (20), Z_{1-5} are divergent contributions; in other words we have set the renormalised couplings κ_{1-5} to zero for simplicity. The anomalous dimensions Z_{λ} etc, and the renormalisation constants for the couplings g , y , C and $(\kappa-1)$, start with tree-level values of 1. (The slightly non-standard definition of Z_{κ} is once again to make our results correspond more closely with those of Ref. [34].) The anomalous dimensions for the gauge-multiplet fields and hence the gauge β -functions are the same as in the standard $\mathcal{N} = 1$ theory. Since our gauge-fixing term in Eq. (14) does not preserve supersymmetry, the anomalous dimensions Z_A and Z_{λ} for A_{μ}^a and λ^a are different (and moreover gauge-parameter dependent), as are those (Z_{ϕ} and Z_{ψ}) for ϕ^a and ψ^a . Moreover, neither Z_{ϕ} nor Z_{ψ} coincide with Z_{Φ} , the chiral superfield renormalisation constant.

We have assigned the same coupling y to all the three-point interactions; for instance, both $d^{abc}\phi^a\phi^b\phi^c$ and $d^{0bc}\phi^0\phi^b\phi^c$. This is by no means guaranteed a priori. From the non-renormalisation theorem, one expects

$$Z_y = Z_\Phi^{-\frac{3}{2}} \quad (21)$$

and so consistency requires Z_Φ and Z_{Φ^0} to be equal. This is arranged by a judicious choice of Z_κ (a change in Z_κ alters Z_Φ while leaving Z_{Φ^0} unchanged).

At one loop we find, writing $Z^{(n)}$ for the n -loop contribution to Z ,

$$\begin{aligned} Z_\phi^{(1)} &= [-N'y\bar{y} + 2g^2(1 - \alpha)N]L, \\ Z_\psi^{(1)} &= [-N'y\bar{y} - 2g^2(1 + \alpha)N]L, \\ Z_F^{(1)} &= -N'y\bar{y}L, \\ Z_y^{(1)} &= -\frac{3}{2}Z_\Phi^{(1)}, \\ Z_\Phi^{(1)} &= [-N'y\bar{y} + 4g^2N]L, \\ Z_g &= 1 - 2g^2NL, \end{aligned} \quad (22)$$

where (using dimensional regularisation with $d = 4 - \epsilon$)

$$L = \frac{1}{16\pi^2\epsilon} \quad (23)$$

and

$$N' = N + \frac{4}{N\kappa}(1 - \kappa). \quad (24)$$

The remaining renormalisation constants will not be required but the one-loop results can be found in Ref. [38]. The difference between Z_Φ and Z_ϕ , Z_ψ is due solely to the choice of a non-supersymmetric gauge; the gauge-independent terms are the same, and since there are no gauge interactions for the $U(1)$ fields anyway, we have

$$Z_{\Phi^0} = Z_{\phi^0} = Z_{\psi^0}. \quad (25)$$

We now choose

$$Z_\kappa^{(1)} = -\frac{4g^2N\kappa}{\kappa - 1} + \frac{y\bar{y}N(\kappa - 2)}{\kappa - 1} - \frac{2y\bar{y}(2\kappa^2 - \kappa - 1)}{N\kappa^2} \quad (26)$$

which guarantees that Z_{Φ^0} and Z_Φ match at one loop.

We have now dealt with the majority of the renormalisations of fields and couplings. The remaining non-linear renormalisations of λ , F and \bar{F} are largely determined in order to cancel C -dependent divergences; though as we have emphasised, a non-linear renormalisation of F and \bar{F} is required in the usual $\mathcal{N} = 1$ ($C = 0$) case. The precise forms of these non-linear renormalisations are not required for our computation, as we shall explain; and will therefore be omitted, though once again they can be found (at one loop) in

Ref. [38]. We then found in Ref. [38] that C is unrenormalised at one loop, i.e. $Z_C^{(1)} = 0$. Our main interest is in determining whether the C parameter remains unrenormalised at the two loop level. To this end, the simplest approach appeared to be to focus on the $\bar{y}C^{\mu\nu}f^{abc}\partial_\mu\bar{\phi}^a\partial_\nu\bar{\phi}^b\bar{\phi}^c$ term. There are two reasons for this. Firstly, fermion calculations frequently produce the quantities

$$\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}, \quad \epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma}, \quad (27)$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the four-dimensional alternating symbol and $\sigma_{\rho\sigma}, \bar{\sigma}_{\rho\sigma}$ are defined in Eq. (11). In exactly four dimensions we have

$$\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = 2\sigma^{\mu\nu}, \quad \epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_{\rho\sigma} = -2\bar{\sigma}^{\mu\nu} \quad (28)$$

(i.e. $\sigma_{\rho\sigma}, \bar{\sigma}_{\rho\sigma}$ are selfdual and anti-self-dual respectively) but it is not clear if these identities remain true away from four dimensions and can therefore be used in the context of dimensional regularisation. Choosing a purely bosonic interaction seems likely to reduce the numbers of appearances of these quantities, and in fact we shall find it never appears in our calculation (though we do meet the quantity $\epsilon^{\mu\nu\rho\sigma}C'_{\rho\sigma}$, which we shall discuss shortly). Secondly, this interaction contains no auxiliary field F and hence is unaffected by any non-linear renormalisation of the auxiliary field \bar{F} which would otherwise also need to be determined in order to fix the value of Z_C .

As we mentioned earlier, we shall only consider the Yukawa-dependent terms in the two-loop renormalisation constant for C . The main reason for setting aside the remaining graphs (which would contribute g^4 terms at this loop order) is their sheer number, namely around a hundred; but there are other technical reasons which we shall discuss in due course. The two-loop diagrams which contribute to the $(y\bar{y})^2$ terms in the renormalisation of $\bar{y}C^{\mu\nu}f^{abc}\partial_\mu\bar{\phi}^a\partial_\nu\bar{\phi}^b\bar{\phi}^c$ are depicted in Fig. 1 and those which contribute to the $g^2y\bar{y}$ terms are shown in Figs. 2, 3. The diagrams in Fig. 1 give a vanishing contribution on grounds of symmetry (in fact we have omitted several diagrams from Figs. 2, 3 which give vanishing contributions for similar reasons, in addition to diagrams which give no logarithmic divergences after subtraction of divergent subdiagrams). For instance, diagrams with two $\bar{\phi}$ (and not $\partial\bar{\phi}$) lines emerging from the same vertex or connected by an auxiliary chiral propagator are zero by symmetry. The divergent contributions from the diagrams (a)-(p) in Fig. 2, 3 are denoted by $G_1 - G_{16}$ respectively and listed below (we perform subtractions of subdivergences on a diagram-by-diagram basis, so that individual results are purely

local):

$$\begin{aligned}
G_1 &= 2L^2 N_1 (1 - \frac{3}{4}\epsilon), \\
G_2 &= 2L^2 \left[N^2 + 2 \left(\frac{6}{\kappa} - 4 \right) + 8K_1 \right] (1 - \frac{1}{2}\epsilon), \\
G_3 &= \frac{1}{2}L^2 \left[N^2 + 2 \left(\frac{6}{\kappa} - 4 \right) + 8K_1 \right] \epsilon, \\
G_4 &= \alpha L^2 N_1 X, \\
G_5 &= -L^2 N_2 X, \\
G_6 &= \frac{3}{4}L^2 N_2 (1 - \frac{7}{12}\epsilon), \\
G_7 &= \frac{1}{4}L^2 N_2 \epsilon, \\
G_8 &= -\alpha L^2 N_1 X, \\
G_9 &= 0, \\
G_{10} &= \frac{1}{4}L^2 N_1 \epsilon, \\
G_{11} &= \frac{1}{2}L^2 N_1 [3 + \frac{1}{2}(\alpha - \frac{1}{2})\epsilon], \\
G_{12} &= -\frac{1}{2}L^2 N_1 [3 + \frac{1}{2}(\alpha - \frac{7}{2})\epsilon], \\
G_{13} &= -2L^2 N_1 X, \\
G_{14} &= -L^2 \left[N^2 + 4 \left(\frac{4}{\kappa} - 3 \right) + 16K_1 \right] X, \\
G_{15} &= \frac{1}{2}L^2 \left[N^2 + 4 \left(\frac{4}{\kappa} - 3 \right) + 16K_1 \right] \epsilon, \\
G_{16} &= -\frac{3}{4}L^2 N_2 (1 - \frac{17}{36}\epsilon) \\
&\quad - \frac{2}{3}L^2 \left[N^2 + \left(\frac{14}{\kappa} - 10 \right) + 12K_1 \right] \epsilon,
\end{aligned} \tag{29}$$

where

$$X = 1 - \frac{1}{4}\epsilon \tag{30}$$

and

$$\begin{aligned}
N_1 &= NN' = N^2 + 4 \left(\frac{1}{\kappa} - 1 \right), \\
N_2 &= N^2 + 4 \left(\frac{2}{\kappa} - 1 \right), \\
K_1 &= \frac{1}{N^2} \left(2 - \frac{3}{\kappa} + \frac{1}{\kappa^3} \right),
\end{aligned} \tag{31}$$

with N' as defined in Eq. (24). Some group identities used in deriving these results are listed in the Appendix. Note that the “deformed” vertex in diagram (p) contains contributions from both terms in the 9th line of Eq. (5). In the case of the majority of diagrams, the only property we have assumed for $\epsilon^{\mu\nu\rho\sigma}$ and $C^{\mu\nu}$ is that they are (totally) antisymmetric

tensors. However, in deriving the result for diagram (n) we have also assumed that the identity

$$\epsilon^{\mu\nu\rho\sigma}C_{\rho\sigma} = 2C^{\mu\nu} \quad (32)$$

(which is valid in four dimensions—i.e. $C_{\mu\nu}$ is self-dual) remains true in $d = 4 - \epsilon$ dimensions. This seems to be a natural requirement and somewhat in the spirit of dimensional reduction; in any case, we shall discuss this choice in more detail later. We then readily find

$$\sum_1^{16} G_i = 0, \quad (33)$$

which implies

$$\left(Z_C Z_y Z_\phi^{\frac{3}{2}} \right)_{\text{pole, } y\text{-dependent}}^{(2)} = 0. \quad (34)$$

In order to determine Z_C we need to know Z_y and Z_ϕ through two loops. The two-loop result for $Z_\phi^{(2)}$ is given in Ref. [39] and translates (in our conventions) to

$$\begin{aligned} Z_{\phi, \text{DREG}}^{(2)} = & L^2(y\bar{y})^2 \left[-N_1^2 + \frac{2}{\kappa N^2}(N^2 - 4) + \frac{4}{\kappa^2 N^2}(N^2 - 2) + \frac{4}{\kappa^4 N^2} \right] (1 - \tfrac{3}{4}\epsilon) \\ & + L^2 y\bar{y} g^2 \left[2N_1(2 + \alpha) - \frac{8}{\kappa} - \left(N^2 - 4 + \frac{2}{\kappa} \right) \epsilon \right] + \dots \end{aligned} \quad (35)$$

where the ellipsis indicates g^4 terms which we shall not require. This result is computed (as the notation indicates) using dimensional regularisation (DREG) (where the number of gauge fields becomes d in d dimensions), while for a supersymmetric calculation we should be using dimensional reduction (DRED) (where the number of gauge fields is maintained as exactly four, even in d dimensions). The difference between our $Z_\phi^{(2)}$ and $Z_{\phi, \text{DREG}}^{(2)}$ resides (as far as the y -dependent terms are concerned) in the two diagrams in Fig. 4 which contribute to Z_ϕ —specifically in the two σ matrices contracted by the gauge propagator. We find

$$Z_\phi^{(2)} - Z_{\phi, \text{DREG}}^{(2)} = \frac{2}{\kappa} L^2 g^2 y\bar{y} \epsilon. \quad (36)$$

This conclusion may be confirmed using the results of Ref. [40]. This reference presented results for Z_ϕ at two loops computed using DRED, together with expressions for the differences between β functions computed using the two schemes. However we cannot use these DRED results directly because they are presented in the Feynman background gauge and we require results for a general conventional gauge. Luckily, the difference between DRED and DREG for the $g^2 y\bar{y}$ terms is gauge-independent. Combining Eqs. (35), (36) we have

$$\begin{aligned} Z_\phi^{(2)} = & L^2(y\bar{y})^2 \left[-N_1^2 + \frac{2}{\kappa N^2}(N^2 - 4) + \frac{4}{\kappa^2 N^2}(N^2 - 2) + \frac{4}{\kappa^4 N^2} \right] (1 - \tfrac{3}{4}\epsilon) \\ & + L^2 y\bar{y} g^2 \left[2N_1(2 + \alpha) - \frac{8}{\kappa} - (N^2 - 4)\epsilon \right]. \end{aligned} \quad (37)$$

The result for $Z_\Phi^{(2)}$ may be extracted from Ref. [41]

$$Z_\Phi^{(2)} = L^2(y\bar{y})^2 \left[-N_1^2 + \frac{2}{\kappa N^2}(N^2 - 4) + \frac{4}{\kappa^2 N^2}(N^2 - 2) + \frac{4}{\kappa^4 N^2} \right] (1 - \frac{3}{4}\epsilon) + L^2 y\bar{y} g^2 (N^2 - 4)(2 - \epsilon). \quad (38)$$

As a check the double poles may also be obtained from the one-loop results using

$$Z_{\phi\text{doublepole}}^{(2)} = \frac{1}{2} \left\{ \left(Z_\phi^{(1)} \right)^2 + \left[Z_y^{(1)} y \frac{\partial}{\partial y} + Z_{\bar{y}}^{(1)} \bar{y} \frac{\partial}{\partial \bar{y}} + Z_g^{(1)} g \frac{\partial}{\partial g} + Z_\kappa^{(1)} (\kappa - 1) \frac{\partial}{\partial \kappa} \right] Z_\phi^{(1)} \right\} \quad (39)$$

with a similar result for Z_Φ .

Using Eqs. (21), (22), (37), (38) we can conclude that

$$Z_C = O(g^4) \quad (40)$$

up to two loops.

4 Conclusions

We have found that Z_C is unrenormalised through two loops, as far as the Yukawa-dependent contributions are concerned; this seems a strong indication that Z_C is completely unrenormalised at this order. Nevertheless it would be reassuring to compute Z_C in full, so we shall now discuss further our choice of prescription Eq. (32) for $C^{\mu\nu}$ and also the feasibility of completing the calculation by including the remaining g^4 terms.

In four dimensions we have the identity

$$\begin{aligned} \sigma^\mu \bar{\sigma}^\nu \sigma^\rho &= -\eta^{\mu\nu} \bar{\sigma}^\rho - \eta^{\nu\rho} \bar{\sigma}^\mu + \eta^{\mu\rho} \bar{\sigma}^\nu + \epsilon^{\mu\nu\rho\sigma} \bar{\sigma}_\sigma, \\ \bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho &= -\eta^{\mu\nu} \sigma^\rho - \eta^{\nu\rho} \sigma^\mu + \eta^{\mu\rho} \sigma^\nu - \epsilon^{\mu\nu\rho\sigma} \sigma_\sigma. \end{aligned} \quad (41)$$

By contracting Eqs. (41) with $\bar{\sigma}_\rho$, σ_ρ it is easy to derive the identities

$$\epsilon^{\mu\nu\rho\sigma} \sigma_{\rho\sigma} = (d-2) \sigma^{\mu\nu}, \quad \epsilon^{\mu\nu\rho\sigma} \bar{\sigma}_{\rho\sigma} = -(d-2) \bar{\sigma}^{\mu\nu}, \quad (42)$$

In order to reconcile Eqs. (42), (32) it seems that we must abandon the identity Eq. (10) (or at least modify it away from two dimensions). The reason that we have not had to confront this issue so far in our calculation might be that to leading order one can prove the invariance of the chiral part of Eq. (5) under Eq. (9) without assuming anything about $C^{\mu\nu}$ other than its antisymmetry; in particular the relation Eq. (10) between $C^{\mu\nu}$ and $\sigma^{\mu\nu}$ is not required. Of course an alternative would be to impose a different prescription to Eq. (32), for instance one analogous to Eq. (42); however this would change the simple pole term for Fig. 2(k) and would introduce (amongst other terms) a $\frac{1}{\kappa^3}$ term into Z_C . On the other hand, it can be seen that there is no diagram contributing to the renormalisation of (for instance) $C^{\mu\nu} d^{ABC} e^{AD} F_{\mu\nu}^D \bar{\lambda}^B \bar{\lambda}^C$ in Eq. (5) which could contain $\frac{1}{\kappa^3}$ dependence; nor does it appear in Z_ϕ or Z_y . Therefore it cannot occur in Z_C .

Clearly we would like to see whether Z_C really does vanish completely through two loops by examining the remaining g^4 -type diagrams. However the complete proof of the $\mathcal{N} = \frac{1}{2}$ invariance of Eq. (5) requires use of Eq. (13) which in turn depends on Eq. (10). It therefore seems that it may be difficult to find a consistent definition for $C^{\mu\nu}$ which will maintain the complete invariance of Eq. (5) in general d dimensions, and this means that we may not expect to obtain an unambiguous answer for the g^4 contribution to Z_C within dimensional regularisation. Indeed, although it does not appear that the potentially ambiguous quantity $\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$ arises in any of the g^4 graphs, we do find ourselves obliged to simplify contractions of the form $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\rho\sigma\alpha\beta}$. In exactly four dimensions we have

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\kappa\lambda\alpha\beta} = 4!\delta^\mu_{[\kappa}\delta^\nu_{\lambda}\delta^\rho_{\alpha}\delta^\sigma_{\beta]}. \quad (43)$$

If we assume that this result remains true away from four dimensions then we obtain

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\rho\sigma\alpha\beta} = 2(d-2)(d-3)\delta^\mu_{[\alpha}\delta^\nu_{\beta]}, \quad (44)$$

but once again the consistency with Eqs. (42), (32) is a moot point.

A possible alternative approach to the two-loop calculation could be the use of differential regularisation [42] which enables one to work in exactly four dimensions. However if one accepts that the results we have so far obtained are a strong indication of the non-renormalisation of Z_C at two loops, then a more fruitful approach may be to seek a general proof of the result to all orders. One may speculate that the non-renormalisation of the non-anticommutativity parameter may follow from some kind of analogue of the Slavnov-Taylor identities. This might be somewhat difficult to see in this component formulation but might be more transparent in the superspace formalism combined with the background field formalism [22, 23, 34, 37]; where the result would be more comparable to a simple Ward identity, due to the manifest supersymmetry and gauge invariance in this case [43].

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5 Appendix

Identities for $SU(N)$ useful for simplifying the divergent contributions are [44]

$$\begin{aligned}
\text{tr}[\tilde{D}^a \tilde{D}^b] &= \frac{N^2 - 4}{N} \delta^{ab}, & \text{tr}[\tilde{D}^a \tilde{D}^b \tilde{D}^c] &= \frac{N^2 - 12}{2N} d^{abc}, \\
\text{tr}[\tilde{F}^a \tilde{F}^b \tilde{D}^c] &= \frac{N}{2} d^{abc}, & \text{tr}[\tilde{F}^a \tilde{D}^b \tilde{D}^c] &= i \frac{N^2 - 4}{2N} f^{abc}, \\
\text{tr}[\tilde{D}^a \tilde{F}^b \tilde{D}^d \tilde{D}^c \tilde{D}^d] &= i \frac{(N^2 - 12)(N^2 - 4)}{4N^2} f^{abc}, \\
\text{tr}[\tilde{D}^a \tilde{D}^b \tilde{D}^d \tilde{D}^c \tilde{F}^d] &= i \frac{(N^2 - 4)^2}{4N^2} f^{abc}, \\
\text{tr}[\tilde{D}^a \tilde{F}^b \tilde{F}^d \tilde{F}^c \tilde{D}^d] = \text{tr}[\tilde{F}^a \tilde{F}^b \tilde{F}^d \tilde{D}^c \tilde{D}^d] &= -i \frac{(N^2 - 4)^2}{4N^2} f^{abc}.
\end{aligned} \tag{45}$$

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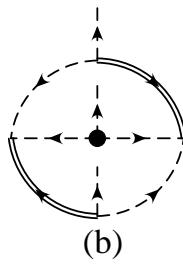
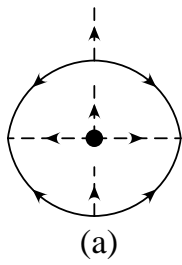


Figure 1: Two-loop $(y\bar{y})^2$ graphs

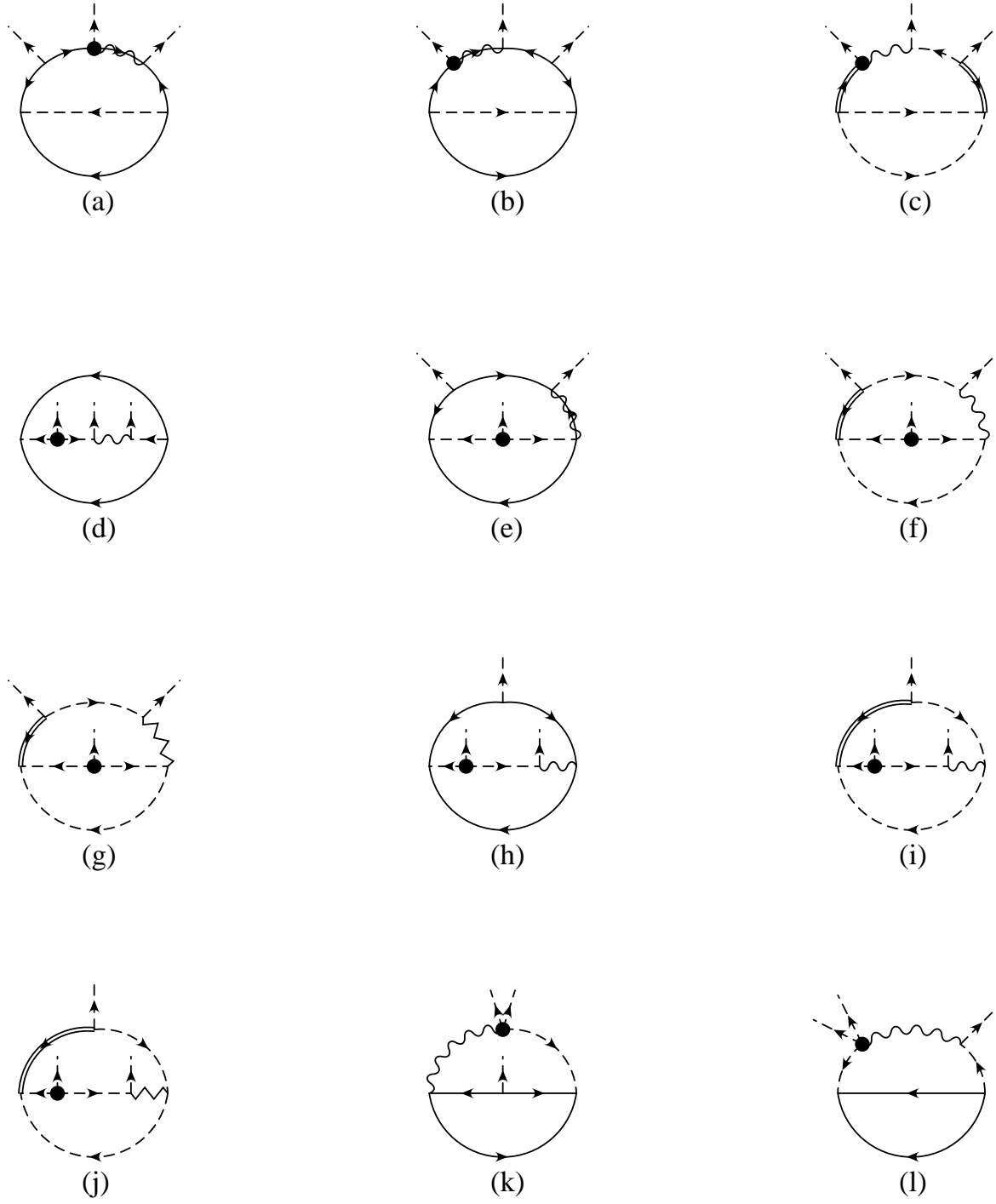


Figure 2: Two-loop $g^2 y \bar{y}$ graphs

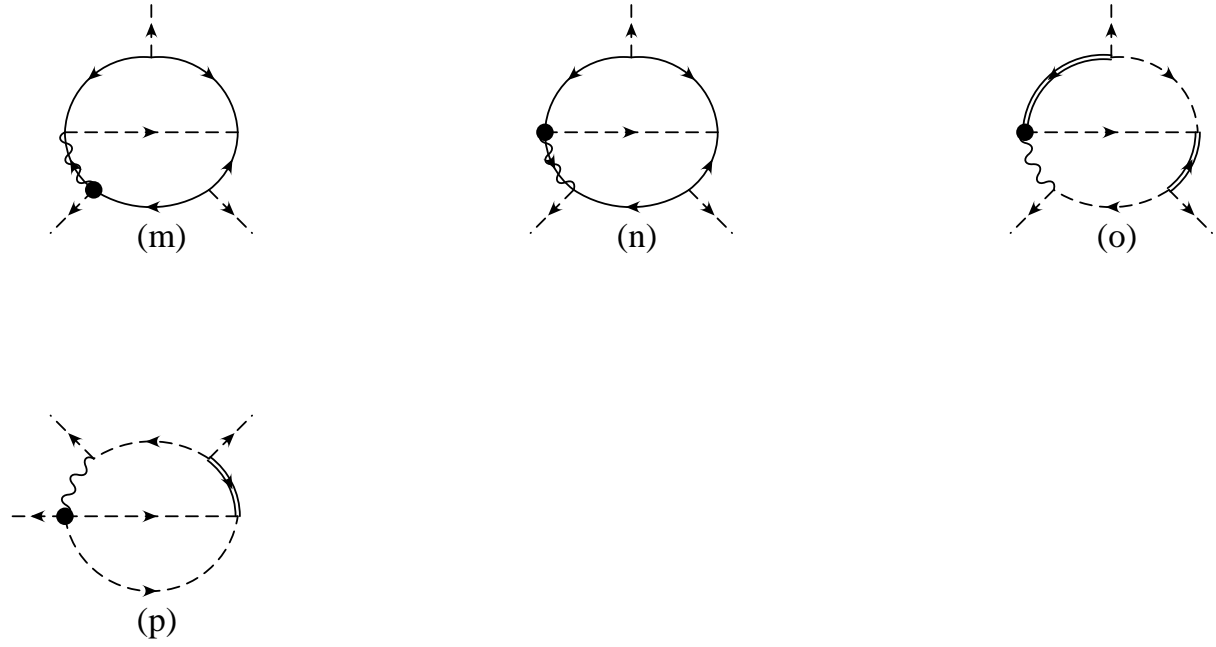


Figure 3: Two-loop $g^2 y \bar{y}$ graphs (continued)

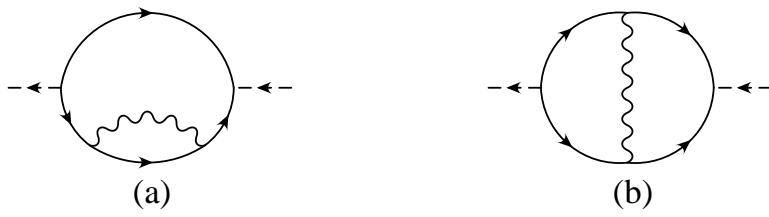


Figure 4: Graphs for Z_ϕ contributing to DREG/DRED difference